mass velocity of fluid based on sectional area of empty tube in the direction of fluid flowing

$\beta = l_p/D_p$
$\gamma = l_s/D_p$
$\varphi = l_v/D_p$
ϵ = fraction void

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Behavior of Suspended Particles in a Turbulent Fluid

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Heat and mass transfer and coagulation are considered as related to the mean-square relative velocity between particle and fluid $\overline{u_R^2}$ and eddy diffusion as described by the mean-square particle displacement, $\overline{X_P^2}$. The mathematical methods used are similar to those employed in the early calculations of the Brownian motion.

The mean-square relative velocity is obtained as a function of particle characteristics, intensity of turbulence, and a fluid correlation coefficient. In the limiting case of equal particle and fluid density $\overline{u_R^2} = 0$, and for very heavy particles $\overline{u_R^2} \to \overline{u_F^2}$.

A general expression for the eddy diffusivity is obtained which reduces to the same form as that of the fluid for the stationary state. However, the correlation coefficient to be used in the calculations depends on $\overline{u_R^2}$. As a first approximation, it can be assumed that at a sufficiently long time from the introduction of the particles, fluid and particle diffusivities are equal.

For short times after injection, the particle spread may be much less than that of the fluid. An illustrative calculation for the initial spreading of a jet of suspended particles is offered. In all cases an effort is made to organize the available experimental data within the framework of the theory.

The behavior of dilute suspensions of small particles in a turbulent fluid is of interest in agitation, fuel injection, erosion, silt transport, and cloud formation. During these processes a number of related effects may occur including (1) heat and mass transfer to the suspended particles, (2) eddy diffusion of the dispersed material, and (3) coagulation or collision.

Practically all the available experimental data are concerned with the second effect, eddy diffusion. The com-

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bined settling and diffusion of particles in water has been studied by Rouse (*11*) and Kalinske and Pien (*7*). Longwell and Weiss (*9*) have studied initial spreading of fuel droplets injected into turbulent air, and Friedlander (*3*) has investigated the deposition of suspended particles from turbulent air. Theories of particle diffusion have been derived in an interesting monograph by Tchen (*14*) and, more recently, in a study by Soo (*12*). A theory of turbulent coagulation has been advanced by Frisch (*5*).

This paper considers heat and mass transfer and coagulation as related to the relative velocity between particle and fluid, and eddy diffusion as described by the mean-square particle displacement. The mathematical methods used were originally developed for the calculation of the Brownian motion (*16*) and random noise in electrical devices. An effort is made to organize the available experimental data within the framework of the theory, with the emphasis, in general, on quantitative explanation rather than mathematical rigor.

As is common in the treatment of turbulent phenomena, handling the problem mathematically requires the consideration of simple systems. In this case, a homogeneous turbulent field containing particles smaller than the microscale of the fluctuating motion being assumed, the results will be best applied to such regions as the center of a pipe or to the atmosphere and with more difficulty to agitated tanks or boundary layers; however, even in the more complicated cases qualitative conclusions are possible.

MEAN-SQUARE RELATIVE-VELOCITY AND TRANSFER RATES

Rates of heat and mass transfer to dilute suspensions of small particles in a turbulent fluid are of determining influence in such phenomena as the growth of organisms in stirred tanks, the evaporation of droplets in spray drying and in fuel injection, and the formation of clouds. The usual theoretical or empirical correlations for transfer rates to small particles apply to spheres in an *infinite* body of fluid which moves past the particle with a *constant* velocity. If the relative velocity between a particle and the fluid around it is known, the transfer coefficient can be obtained.

Clearly, however, the requirements of a large body of fluid moving at constant velocity past the particle are not fulfilled in the case of particles suspended in a turbulent fluid. In the atmosphere and in large ducts it is often the case that the particles together with their associated thermal or concentration boundary layers are smaller than the Eulerian microscale of turbulence, the distance over which velocity differences are high. Mickelsen (*10*) found that for air flow in an 8-in. duct at velocities ranging from 50 to 160 ft./sec. the Eulerian microscale at the pipe axis was approximately 4,500μ. This is considerably larger than the usual drop size encountered, for example in fuel injection. Thus in this instance there is some justification for assuming that the particle moves in a large, uniform field of flow.

There is, however, little justification for assuming that the velocity past the particle is constant, since it must vary with time for each particle and over all the particles at any time. Therefore the root-mean-square relative velocity between particle and fluid is assumed to represent the characteristic value for transfer calculations. If values of this parameter were available, the mean Reynolds number could be estimated and the Nusselt number obtained from correlations such as that of Friedlander (*4*) for low Reynolds numbers. The transfer coefficient might then be used with a driving force based on the concentration or temperature difference between the bulk of the fluid and the surface of the particle to determine the transfer rate.

Evidently such a calculation hinges on a method for obtaining the root-mean-square relative velocity, which can be estimated in the following manner for a fluid in homogeneous turbulence containing a large number of small, rigid, noninteracting spheres.

The particles have been in the fluid long enough to have reached a steady statistical configuration. They are much smaller than the Eulerian microscale of turbulence but large enough for their Brownian motion to be negligible. (There is no correlation between the turbulent and Brownian fluctuations, and the approach which follows can easily be modified to take both into account.)

Each particle is assumed to be contained within a large packet of fluid, an "eddy," having a velocity u_F and an acceleration a_F. One might picture, for example, a particle of sand in a thimbleful of fluid. Both the fluid velocity and the fluid acceleration vary with time; moreover, the particle may move gradually from packet to packet, and this movement is more pronounced the larger and heavier the particle.

A force balance may now be written on the particle in the x direction, taken perpendicular to the gravitational field. Provided that the Reynolds number $d_P\sqrt{\overline{u_R^2}}\rho/\mu$ is 1 or less, the resistance of the fluid to the particle motion will be governed by Stokes's Law:

$$F_f = -f(u_p - u_F) \qquad (1)$$

This expression is adequate for small particles in gases but may be inaccurate for large accelerations in liquids. [See, for example, Hughes and Gilliland (*6*).]

In addition to the frictional resistance, the particle will be subjected to a "buoyant" force because of the pressure gradient within the packet of fluid. In general, the fluctuating pressure gradients will be related to the fluctuating fluid accelerations through the equations of fluid motion. If the Reynolds number of turbulence is sufficiently high, however, it appears that the frictional terms in these equations may be neglected (*1, 13*). In this case the pressure gradient is related to the acceleration by the expression

$$\frac{\partial p}{\partial x} = -\rho_F a_F \qquad (2)$$

and the buoyant force acting on the particle will be

$$F_b = m_F a_F \qquad (3)$$

Thus the force balance describing the motion of the particles takes the form

$$m_p a_p = m_F a_F - f(u_p - u_F) \qquad (4)$$

or

$$\frac{du}{dt} u_p + \beta u_p = \beta u_F + \gamma \frac{du_F}{dt} \qquad (5)$$

Physically, β^{-1} is a measure of particle inertia; it represents the time necessary for the velocity of a particle injected into a stagnant fluid to reach $1/e$ of its initial value.

As the fluid-particle system is in a steady, homogeneous, and isotropic turbulence, the fluid velocity (and, in a related manner, its acceleration) possesses the following characteristics:

1. The mean value of the fluid velocity taken over all the particles at any instant is zero:

$$\bar{u}_F(t) = 0 \qquad (6)$$

2. There exists some correlation between the values of the fluctuating fluid velocity at any two times t_1 and t_2 such that

$$\overline{u_F(t_1)u_F(t_2)} = \overline{u_F}^2 R(t_2 - t_1) \qquad (7)$$

where

$\overline{u_F^2}$ = mean square fluid velocity summed over all particles

This correlation is finite only for very small values of $t_2 - t_1$. In other words, there is little relation between the velocities of the fluid surrounding the particle at any two times unless these times are close to each other. It should be noted that this correlation coefficient relates the velocities of the fluid in the neighborhood *of the particle* at different times. For a given turbulent fluid it is a function of the size and density of the particle; these determine the "slip" or motion relative to the fluid. In general, the coefficient will vary from approximately the Lagrangian fluid correlation for extremely small particles to the Eulerian time coefficient for large, heavy particles.

Proceeding with the calculation of the mean velocity, upon integrating Equation (5) one obtains

$$u_p = u_{p0}e^{-\beta t} + \beta e^{-\beta t}\int_0^t e^{\beta T} u_F \, dT + \gamma e^{-\beta t}\int_0^t e^{\beta T} a_F \, dT \qquad (8)$$

Integrating the last term by parts, subtracting $u_F - u_{F0}e^{-\beta t}$ from both sides, and rearranging yields

$$u_R - u_{R0}e^{-\beta t} = (1 - \gamma)[(u_{F0}e^{-\beta t} - u_F) + \beta e^{-\beta t}\int_0^t e^{\beta T} u_F \, dT] \qquad (9)$$

Squaring and taking mean values over all particles at time t gives

$$\overline{u_R^2} - 2\overline{u_R u_{R0}}e^{-\beta t} + \overline{u_{R0}^2}e^{-2\beta t} = (1 - \gamma)^2\{\overline{u_{F0}^2}e^{-2\beta t} - 2\overline{u_{F0}u_F}e^{-\beta t} + \overline{u_F^2} + 2\beta e^{-\beta t}\int_0^t e^{\beta T}\overline{u_{F0}u_F(T)} \, dT - 2\beta \int_0^t e^{-\beta(t-T)}\overline{u_F(T)u_F(t)} \, dT + \beta^2 e^{-2\beta t}\overline{\left[\int_0^t e^{\beta T} u_F \, dT\right]^2}\} \qquad (10)$$

From Equation (7)

$$\overline{u_F u_{F0}} = \overline{u_F^2}R(\theta) \qquad (11)$$

and the first integral on the right becomes

$$\overline{I_1} = \overline{u_F^2}\int_0^t e^{\beta\theta}R(\theta) \, d\theta \qquad (12)$$

If $\theta = t - T$, the second integral can be written

$$\overline{I_2} = \overline{u_F^2}\int_0^t e^{-\beta\theta}R(\theta) \, d\theta \qquad (13)$$

To evaluate the last quantity in (10), one writes the mean square of the integral in terms of two independent variables of integration T_1 and T_2:

$$\overline{I_3} = \int_0^t \int_0^t e^{\beta(T_1+T_2)} \cdot \overline{u_F(T_1)u_F(T_2)} \, dT_1 \, dT_2 \qquad (14)$$

This integral may be evaluated by letting $\theta_1 = T_2 - T_1$ and $\theta_2 = T_1 + T_2$. The Jacobian for the coordinate transformation is ½; evaluating the new limits of integration and remembering that R is an even function, one may write

$$\overline{I_3} = \overline{u_F^2}\int_0^t R(\theta_1) \, d\theta_1 \int_{\theta_1}^{2t-\theta_1} e^{\beta\theta_2} \, d\theta_2 = \frac{\overline{u_F^2}}{\beta}\left\{\int_0^t [R(\theta)] \cdot (e^{-\beta\theta+2\beta t} - e^{\beta\theta}) \, d\theta\right\} \qquad (15)$$

In general, the correlation coefficient $R(\theta)$ decreases rapidly with θ. This is a property inherent in the turbulent fluid. Thus integrals of the type of (13) quickly approach a limit, as the effect of the term $e^{-\beta\theta}$ is to make the integrand decrease even more rapidly. Substituting (15), (13), (12), and (11) in (10) and considering the motion as t approaches infinity, one finds that terms with multipliers $e^{-\beta t}$ and $e^{-2\beta t}$ vanish, giving the steady state value

$$\overline{u_R^2} = (1 - \gamma)^2\overline{u_F^2} \cdot \left(1 - \beta \int_0^\infty e^{-\beta\theta}R(\theta) \, d\theta\right) \qquad (16)$$

From this result a number of interesting conclusions may be drawn. When particle and fluid are of equal density, $\gamma = 1$ and $\overline{u_R^2} = 0$, as the particle follows the motion of the fluid. When the particle is extremely heavy, β and γ are small and $\overline{u_R^2}$ approaches $\overline{u_F^2}$. If the particle is very small (β very large) but of a density different from that of the fluid ($\gamma \neq 1$), it is necessary to consider the correlation coefficient only in the neighborhood of $\theta = 0$ since $e^{-\beta\theta}$ decreases so rapidly. Expanding $R(\theta)$ [see Frenkiel (*2*), for example] gives

$$R(\theta) = 1 - \frac{\theta^2}{\lambda^2} + \frac{\theta^4\overline{\left(\frac{da_F}{d\theta}\right)^2}}{24(\overline{u_F^2})^2} - \cdots \qquad (17)$$

Substituting in (16) and integrating yields

$$\overline{u_R^2} \approx 2(1 - \gamma)^2 \frac{\overline{u_F^2}}{\lambda^2\beta^2} \qquad (18)$$

This expression should apply to very small particles in a liquid.

In the general case it is necessary to have values for the correlation coefficient as a function of time. If the particles follow the motion of the fluid approximately, the proper coefficient to use is the Lagrangian correlation. This can be checked by substituting in (15) to see whether $\overline{u_R^2}$ is small. If it is a large percentage of $\overline{u_F^2}$, the Eulerian time coefficient should be used.

There does not seem to be any information available concerning the relationship between these two coefficients although the data on eddy diffusion discussed in the next section indicate that they are probably similar in form. In the absence of more certain information, mean relative velocities may be estimated on the basis of the data of Mickelsen (*10*), who measured turbulence intensity and correlation at the axis for air flow in an 8-in. pipe. On the assumption that the form for the Langrangian coefficient is given by

$$R(\theta) = e^{-a\theta} \qquad (19)$$

the parameter a was found to be 200 sec.$^{-1}$ from Mickelsen's data for $U = 75$ ft./sec. Substituting in (16) and integrating yields

$$\frac{\overline{u_R^2}}{\overline{u_F^2}} = \frac{a}{a + \beta} \qquad (20)$$

In Figure 1 $\overline{u_R^2}/\overline{u_F^2}$ is plotted as function of particle size for air at 20°C. Clearly, for particles larger than about 10μ, "slip" occurs and the mean relative velocity becomes significant. Values for the Nusselt number for mass transfer, a Schmidt group of unity being assumed, are also shown in Figure 1. It is evident that the assumption sometimes made of Nusselt number = 2 (stagnant case) for particles carried along by a turbulent fluid may be in error. The dotted parts of the curves are uncertain because slip occurs (which makes suspect the use of the Lagrangian coefficient) and because the mean relative Reynolds number is greater than 5 (contrary to the assumption of Stokes's Law).

The variation of mean-square velocity among particles of different sizes in the turbulent field will result in collision and, for cases of inelastic impact, in agglomeration or coalescence. A clear picture of this phenomenon can be obtained from a consideration of fog droplets in atmospheric turbulence. In this case scale is very large and the system may be considered to consist of masses of air which contain suspended particles and which move in a random manner. The larger particles, which have high mean-square relative velocities, will tend to sweep out the smaller ones with efficiencies of collision which can be estimated from calculations such as those of Langmuir (*8*). Such an explanation offers a mechanism for rain formation by coalescence of droplets in a turbulent field. The same picture is applicable in agitated systems. Calculations of turbulent coagulation will, of course, depend on information on scale and intensity of turbulence for the system under consideration.

EDDY DIFFUSION OF SUSPENDED PARTICLES

The combined diffusion and settling of suspended particles has been studied by Kalinske and Pien (*7*) and Rouse (*11*), who were particularly interested in problems of silt transport. These investigators studied systems which were in a steady state so that the rate of downward settling was equal to the rate of upward diffusion. Rouse found little variation in diffusivity with particle size; Kalinske and Pien actually used eddy diffusivities determined independently with hydrochloric acid to predict successfully the concentration gradient of fine sand. Friedlander (*3*) studied the deposition of particles from turbulent air on the walls of ducts. Assuming equal particle and fluid diffusivities, he found fair agreement of theory with his data.

Longwell and Weiss (9), on the other hand, were interested in the initial spreading of a fuel spray injected at the center of a turbulent pipe flow. Their theory, which was semiquantitative, indicated that particles with relatively high inertia diffuse less rapidly than fluid. The data which they obtained agreed with their qualitative picture.

In addition to these studies of particle motion, there have been efforts to trace turbulent fluid motion by use of small particles. Thus it is of importance to be able to describe the rate of spreading of the particles at various stages after their introduction into the fluid.

AN ASYMPTOTIC EXPRESSION FOR THE MEAN-SQUARE DISPLACEMENT

The rate of eddy diffusion of the suspended particles can be described by expressing their mean-square displacement $\overline{X_P^2}$ as a function of time. Again a large number of small, noninteracting spheres may be envisioned in a fluid in homogeneous turbulence. The system has reached a steady statistical configuration. The plane $x = 0$, parallel to the gravitational field, intercepts a large number of marked particles at $t = 0$. The distance traveled by one of these particles in the time t is obtained by integrating Equation (5):

$$X_p + \frac{u_p - u_{p0}}{\beta} = \int_0^t u_F(T)\, dT + \frac{\gamma(u_F - u_{F0})}{\beta} \qquad (21)$$

Squaring both sides of the equation and taking averages over all the particles originally in $x = 0$ gives

$$\overline{X_p^2} + \frac{2}{\beta}\int_0^t \overline{u_p(T)[u_p(t) - u_{p0}]}\, dT + \frac{\overline{(u_p - u_{p0})^2}}{\beta^2} = \int_0^t \int_0^t \overline{u_F(T_1)u_F(T_2)}\, dT_1\, dT_2 + \frac{2\gamma}{\beta}\int_0^t \overline{u_F(T)[u_F(t) - u_{F0}]}\, dt + \frac{\gamma^2\overline{(u_F - u_{F0})^2}}{\beta^2} \qquad (22)$$

According to Frenkiel (2), the double integral can be written

$$\overline{I_4} = 2\overline{u_F^2}\int_0^t (t - \theta)R(\theta)\, d\theta \qquad (23)$$

The fact that the correlation coefficient is an even function makes it is easy to show that

$$\int_0^t \overline{u(T)[u(t) - u_0]}\, dT = 0 \qquad (24)$$

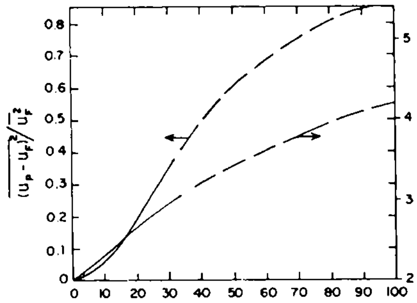

Fig. 1. Predicted relative velocities and mass transfer to particles ($\rho_p = 1$) in 8-in. pipe with $U = 75$ ft./sec.

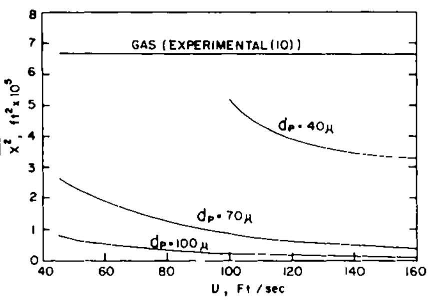

Fig. 2. Predicted spread 4 in. downstream from point of injection for particles ($\rho_p = 1$) in air (20°C.) flowing through an 8-in. pipe.

Substituting in (22), gives

$$\overline{X_p^2} + \frac{2\overline{u_p^2}}{\beta}(1 - R_p(\theta)) = 2\overline{u_F^2}t\int_0^t (t - \theta)R(\theta)\, d\theta + \frac{2\gamma}{\beta}\overline{u_F^2}(1 - R(\theta)) \qquad (25)$$

As t approaches infinity, this becomes

$$\overline{X_p^2} = 2\overline{u_F^2}t\int_0^{t\to\infty} R(\theta)\, d\theta \qquad (26)$$

This expression, which is identical in form with that of the Taylor diffusion equation, was also obtained by Tchen (14), who interpreted it to mean that particles and fluid diffuse at the same rate. However, the correlation coefficient in (26) applies to the fluid velocities *over the path of the particle*. As pointed out above, heavy particles move relatively slowly and cannot follow the fluid eddies which surge around them. Thus the correlation coefficients which should be utilized in (26) range between two extremes: (1) the Lagrangian coefficient for small particles able to follow the fluid and (2) the Eulerian time coefficient for particles which remain almost fixed.

There does not seem to be any information available concerning the relationship between these two coefficients. The data of Kalinske and Pien and those of Rouse for combined eddy diffusion and settling of sand particles in water indicated little variation of the eddy diffusivity with particle size. Thus in their cases, at least, the two types of coefficients were probably similar.

Defining R_p as a correlation coefficient between particle velocities, one may also write the mean-square particle displacement after a sufficiently long times as

$$\overline{X_p^2} = 2\overline{u_p^2}t\int_0^\infty R_p(\theta)\, d\theta = 2\overline{u_F^2}t\int_0^\infty R(\theta)\, d\theta \qquad (27)$$

Thus for heavy particles with low mean-square velocities, the scale as defined by $\int_0^\infty R_p(\theta)\, d\theta$ must be correspondingly large. In other words, heavier particles "remember" for a longer time than lighter ones what has happened to them previously.

This one-dimensional treatment can be extended in the case of nonisotropic turbulence to the Y and Z displacements by defining correlation coefficients in these directions and using the appropriate mean-square velocities. The procedure then is identical with that outlined above, and analogous expressions for the mean-square displacements and velocities will be outlined.

INITIAL SPREADING OF PARTICLES ORIGINALLY AT REST

Although after long periods of time the forms of the equations representing the mean-square particle and fluid displacements are similar, the behavior for short times is quite different. The rate of initial spreading is often of interest when a suspension of particles is injected into a high-velocity gas stream. In the analysis that follows it is assumed that the gas is flowing in a homogeneous, isotropic turbulence. This is approximated near the axis for high-velocity pipe flow. The suspension is introduced at the center of the pipe with the main-stream velocity U in such a way that the characteristics of the turbulence are undisturbed. The initial radial velocity of the particles is zero, and diffusion in the axial direction is negligible in comparison with transport by the fluid.

To find the mean-square lateral displacement of the particles from the axis Equation (5) is integrated with $\gamma = 0$ and with $u_{p0} = 0$, to give

$$X_p = \int_0^t (1 - e^{-\beta(t-T)})u_F\, dt \qquad (28)$$

Squaring and averaging yields

$$\overline{X_p^2} = \int_0^t \int_0^t (1 - e^{-\beta(t-T_1)}) \cdot (1 - e^{-\beta(t-T_2)})\overline{u_F(T_1)u_F(T_2)}\, dT_1\, dT_2 \qquad (29)$$

As in the previous discussion,

$$\overline{u_F(T_1)u_F(T_2)} = \overline{u_F^2}R.$$

In this case, however, R is a function not only of $(T_2 - T_1)$ but also of t, as the system is not in a statistically steady state. For large, heavy particles with low mean-square velocities, the variation with t should not be great; as an approximation, it will be assumed that R is an even function of only the time difference.

Letting $P_1 = t - T_1$ and $P_2 = t - T_2$ and expanding the exponentials and the correlation coefficient $R(T_2 - T_1) = R(P_2 - P_1)$ as in (17) gives

$$\overline{X_p^2} = \overline{u_F^2} \int_t^0 \int_t^0 \left[\beta^2 P_1 P_2 \frac{-\beta^3}{2} \cdot (P_1^2 P_2 + P_1 P_2^2) + \beta^4 \left(\frac{P_1 P_2^3}{6} + \frac{P_1^2 P_2^2}{4} + \frac{P_2 P_1^3}{6} \right) - \frac{\beta^2}{\lambda^2} (P_1 P_2^3 - 2P_1^2 P_2^2 + P_1^3 P_2) + \cdots \right] dP_1\, dP_2 \quad (30)$$

$$= \overline{u_F^2} \left[\beta^2 \frac{t^4}{4} - \beta^3 \frac{t^5}{6} + \left(\frac{5}{72} \beta^4 - \frac{\beta^2}{36\lambda^2} \right) t^6 + \cdots \right] \quad (31)$$

This expression describes the initial spreading of particles issuing with zero velocity from a point source in a turbulent gas. For very short times the correlation coefficient approaches unity, and

$$\overline{X_p^2} \approx \beta^2 \overline{u_F^2} \frac{t^4}{4} \quad (32)$$

The dependence on t^4 is in marked contrast with the behavior of turbulent-fluid particles (*2*), for which

$$\overline{X_F^2} \approx \overline{u_F^2} t^2 \quad (33)$$

Clearly, then, care must be taken in using particles to trace turbulent motion in gases.

Equation (31) also indicates that the effect of the turbulent field does not become important for a while (only in the sixth-order term). Thus as in the case of the fluid, initial spreading of particles should depend little on the correlation coefficient but very much on the intensity of turbulence. This is the same conclusion to which Longwell and Weiss came on a more qualitative basis.

To continue with the calculation of lateral spread information is needed on turbulence intensities and correlation. Mickelsen (*10*), who measured the diffusion of helium injected from a point source at the center of an 8-in. pipe, found that in the range of 50 to 160 ft./sec. the radial turbulent intensity was about 0.032. For any point L downstream, $t = L/U$, and with $\sqrt{\overline{u_R^2}}/U = 0.032$ (31) becomes

$$\overline{X_p^2} = \frac{10^{-3}}{4} \frac{\beta^2 L^4}{U^2} \left[1 - \frac{2}{3} \beta \frac{L}{U} + \left(\frac{5}{18} \beta^2 - \frac{1}{9\lambda^2} \right) \left(\frac{L}{U} \right)^2 + \cdots \right] \quad (34)$$

To estimate the parameter λ (which is taken to be the Lagrangian microscale) from the diffusion data of Mickelsen, one uses the method of Uberoi and Corrsin (*15*), who write

$$\frac{\overline{X_F^2}}{\overline{u_F^2} t^2} = 1 - \frac{1}{6} \frac{t^2}{\lambda^2} + \cdots \quad (35)$$

Thus if $\overline{X_F^2}/\overline{u_F^2} t^2$ is plotted vs. t^2, the slope at $t^2 = 0$ is $-1/6\lambda^2$ and the intercept is unity. In this manner it was found that

$$\frac{1}{\lambda^2} = 18U^2 \quad (36)$$

where λ is in seconds when U is in feet per second. In Figure 2, $\overline{X_p^2}$ is plotted as a function of gas velocity with particle size as the parameter for $L = 4$ in. The experimental data of Mickelsen for the diffusion of helium are also shown.

The curves show that for diffusion of gases or very small particles ($\beta \gg 1/t$), $\overline{X_F^2}$ measured at a given point varies little with the air velocity. This can be explained if it is remembered that for short times

$$\overline{X_F^2} = \overline{u_F^2} t^2 = 10^{-3} L^2 \quad (37)$$

On the other hand, for larger particles ($\beta \ll 1/t$) mean-square displacement at a given point decreases rapidly with mainstream velocity. These conclusions are confirmed at least qualitatively by the experimental data of Longwell and Weiss (*9*) for initial spreading of volatile and nonvolatile fuels.

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NOTATION

a = constant, sec.$^{-1}$
a_F = acceleration, sq. cm./sec.
d_p = particle diameter, cm. or μ
f = resistance coefficient = $3\pi \mu d_p$ in Stokes's region, g./sec.
F_b = buoyant force, dynes
F_f = frictional resistance, dynes
$I_{1,2...}$ = integrals
L = distance downstream from injection point, in.
m_F = mass of fluid occupying same volume as m_p, g.
m_P = mass of particle, g.
p = pressure, dynes/sq. cm.
P_1, P_2 = variables of integration, sec.
R = fluid correlation coefficient, dimensionless
R_P = particle correlation coefficient, dimensionless
t = time
T, T_1, T_2 = variables of integration, sec.
u_F = fluid velocity, cm./sec.
u_p = particle velocity, cm./sec.
u_{F0} = initial fluid velocity, cm./sec.
u_{p0} = initial particle velocity, cm./sec.
u_R = $u_p - u_F$, cm./sec.
U = pipe mainstream velocity, cm./sec.
x, y, z = coordinate axes
X_p = particle displacement in x direction, cm.
X_F = fluid displacement in x direction, cm.
Y, Z = displacements in y and z directions, respectively, cm.
β = f/m_p, sec.$^{-1}$
γ = $m_F/m_P = \rho_F/\rho_P$, dimensionless
$\theta, \theta_1, \theta_2$ = variables of integration, sec.
λ = microscale, sec.
μ = fluid viscosity, g./(cm.)(sec.)
ρ_F = fluid density, g./cc.
ρ_P = particle density, g./cc.

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